CHAPTER VIII – TN 23: SENSITIVITY ANALYSIS OF LAND USE ALLOCATION MODEL

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ABSTRACT

It has become increasingly apparent that the sensitivity of linear programming parameters defining a given allocation problem is a weakness of the approach. This paper presents the results of a sensitivity analysis performed on a linear programming land use allocation model. In the problem treated the objective function, defined as the total profit received from single or multiple use of four hypothetical forestry lands is first maximized. Then parametric programming is used to investigate the effect of coefficient changes on the optimal solution.

The utility of having a set of alternatives in a planning perspective is emphasized throughout. In particular, the example allows one to point out the following considerations about the solution that was initially found:

(1) A shift from multiple use to single use results if there is a 3 percent change in one of the profit coefficients.

(2) No great loss in total profit results if a certain activity not in the basis is introduced.

(3) A small change in the opportunity costs used to determine the price for using a unit of an outdoor recreation activity causes a change of policy.

INTRODUCTION

Over the years a steady increase in the use of our natural resources for recreation purposes has resulted in a large amount of literature dealing with estimating use, and valuing benefits, of outdoor recreation sites. Many of the quantitative techniques used to estimate recreation use and analyze outdoor recreation data in general have been drawn from the realm of statistics. These include regression analysis, factor analysis, analysis of variance, cluster analysis, and other multivariate methods.

Relatively recently, an optimization technique, linear programming (LP), has been applied to analyze outdoor recreation data. Meir (1968) developed a linear programming model which incorporates in a mathematical framework many of the factors relevant to the planning of recreation land acquisition expenditures. Tadros and Kalter (1971) worked out a spatial allocation model for projecting outdoor recreation demand which takes the form of a linear programming solution. Manning (1971) dealt with multiple use of forest land in a linear programming context. Saitta and Schmedemann (1972) used linear programming to maximize the number of new visitors to a park on a peak weekend. However, none of these papers dealt with a very important aspect of linear programming, which is sensitivity analysis of the input coefficients used (analysis of how the LP solution changes with changes in the input coefficients).

So, it is the purpose of the present paper to report on the results of a sensitivity analysis performed on a modification of Manning's (1971)model. It should be recognized that once a solution to a linear programming problem has been determined, the cost of sensitivity analysis compared to the cost of finding the solution is usually small.

A LAND USE ALLOCATION MODEL

Consider four pieces of forest land, each of which can be used for one or more different purposes: (1) timber production, (2) agricultural production, (3) recreation, and (4) multiple use. The assumptions concerning the use of the lands are: (1) the lands are already in public ownership, and (2) multiple uses (recreation and forestry use) are compatible.

Information that is known about the lands and various products is as follows:

(1) The production capabilities (Table 1).

(2) The resource availability for production (Table 2).
(3) The prices of agricultural and forestry products (Table 3).
(4) The prices, as derived by using opportunity cost of recreation products (Table 4) as defined by Equations (A.2) - (A.4) in the Appendix.
(5) The resource requirements of each activity (Table 5).

TABLE 1: PRODUCTION CAPABILITIES PER YEAR, PER ACRE OF FOUR HYPOTHETICAL LANDS*

Use		Production of Land			
		A	B	C	D
Timber	White Spruce	0.56	0.40	0.70	-
	Poplar	-	-	-	1.00
Agriculture	Barley	7.00	-	-	-
	Potato	-	105.00	-	70.00
	Wheat	-	-	20.00	-
Recreation	Hunting .	11.00	-	-	3.00
	Camping	-	60.00	-	-
	Fishing	-	-	81.00	-
Multiple-Use	White Spruce	0.56	0.12	0.70	-
	Poplar	-	-	-	1.00
	Hunting	11.00	-	-	3.00
	Camping	-	20.00	-	-
	Fishing	-	-	81.00	-

- is zero production.
* These production capabilities were determined using the Canada Land Inventory method. Timber products are measured in cords, agricultural products in bushels except potato, which is measured in hundred-weights, and recreation "products" in visitor-days.

TABLE 2: RESOURCE AVAILABILITY FOR PRODUCTION OF THE FOUR HYPOTHETICAL LANDS

Resources	Limits on	Resources
Capital	$7,000.00	
Labour	700.00	person-days
Land-A	300.00	acres
Land-B	80.00	acres
Land-C	268.00	acres
Land-D	1,046.00	acres

TABLE 3: MARKET PRICES OF TIMBER AND AGRICULTURAL PRODUCTS

Product	Price
White Spruce	$5.60 / cord
Poplar	$1.34 / cord
Barley	$1.15 / bushel
Wheat	$1.25 / bushel
Potato	$1.25 / hundred-weight

Sources: (1) G. Manning op. cit. (1971)
(2) New Brunswick Department of Agriculture and Rural Development, AGRICULTURAL STATISTICS, 1971, Fredericton, New Brunswick.

TABLE 4: OPPORTUNITY COSTS FOR RECREATION USE OR MULTIPLE-USES OF THE FOUR HYPOTHETICAL LANDS

Land	Use	Opportunity Cost, $
A	Hunting	0.47
	White spruce and hunting	0.36
B	Camping	2.17
	White spruce and camping	1.56
C	Fishing	0.26
	White spruce and fishing	0.06
D	Hunting	4.95
	Poplar and hunting	4.30

TABLE 5: GROSS RETURN, TOTAL COST AND PROFIT OF PRODUCTION PER ACTIVITY PER ACRE OF THE FOUR HYPOTHETICAL LANDS

Land	Activity acre	Product	Gross Return(1),$	Capital Cost(2),$
A	X1	White Spruce	3.14	3.00
	X2	Barley	8.05	12.00
	X3	Hunting	5.14	3.00
	X4	Multiple-Use:		
		White Spruce		1.00
			7.14	
		Hunting		2.00
B	X5	White Spruce	2.24	1.00
	X6	Potato	131.25	175.00
	X7	Camping	130.20	30.00
	X8	Multiple-Use:		
		White Spruce		0.20
			31.84	
		Camping		10.00
C	X9	White Spruce	3.92	3.00
	X10	Wheat	25.00	10.00
	X11	Fishing	21.00	1.00
	X12	Multiple-Use:		
		White Spruce		2.00
			8.92	
		Fishing		1.00
D	X13	Poplar	1.34	0.50
	X14	Potato	87.50	175.00
	X15	Hunting	14.84	3.00
	X16	Multiple-Use:		
		Poplar		0.10
			14.34	
		Hunting		3.00

TABLE 5 (contd)

Land	Activity acre	Labour (2) Person-days	Cost,$	Total Cost(3),$	Profit(4),$
A	X1	0.10	2.00	5.00	- 1.86
	12	0.30	6.00	18.00	- 9.95
	X3	0.20	4.00	7.00	- 1.86
	X4				
		0.10	2.00		
				7.00	0.14
		0.10	2.00		
B	X5	0.05	1.00	2.00	0.24
	X6	1.50	30.00	205.00	-73.75
	X7	5.00	100.00	130.00	0.20
	X8				
		0.01	0.20		
				30.40	1.44
		1.00	20.00		
C	X9	0.25	5.00	8.00	- 4.08
	X10	0.25	5.00	15.00	10.00
	X11	0.50	10.00	11.00	10.00
	X12				
		0.10	2.00		
				9.00	- 0.08
		0.20	4.00		
D	X13	0.05	1.00	1.50	- 0.16
	X14	0.25	5.00	180.00	-92.50
	X15	0.60	12.00	15.00	- 0.16
	X16				
		0.30	0.60		
				13.70	0.64
		0.50	10.00		

(1) The gross return per activity per acre is equal to the unit value of the product (as given in Table 3 and Table 4) multiplied by the quantity of the product per acre (as given in Table 1).

(2) The capital cost per acre and the labour cost per person-day are believed to be "reasonable" estimates. However, when a real situation is dealt with, people knowledgeable in these fields should be consulted in order to make the estimates as realistic as possible. The labour cost was calculated at $20. per person-day.

(3) The total cost is the sum of capital cost and labour cost.

(4) The profit is equal to the gross return minus the total cost per activity per acre.

The objective is to determine the production levels of the activities (uses) within the limitations of the resources, such that total profit is a maximum.

Utilizing the information presented in Tables 2 and 5, one can formulate the linear programming problem as follows:

Maximize CX

subject to $AX \leq b$, $X(j) \geq 0$; j=1,16

WHERE C = The profit coefficient vector as defined in the ninth column of Table 5.

X = The activity vector as defined in the first column of Table 5.

A = The activity coefficient matrix as defined in the fifth and sixth columns of Table 5.

b = The requirement vector as defined in Table 2.

The problem formulated above was solved by the Simplex method (see Reference 26). The solution and the levels of resources used are summarized in Table 6. The maximum total profit obtained is $3,462 per annum. The policy that results in this maximum total profit is to utilize all 80 acres of land-B for white spruce production and camping; all 268 acres of land-C for wheat production; 1042 acres of land-D for poplar production and hunting, and not to use land-A at all.

TABLE 6: LEVELS OF ACTIVITIES AND CONSTRAINTS AT OPTIMAL SOLUTION

	Activity	Level	Constraint	Level
X8	White spruce production & camping land-B	80 acres	Capital	$6,374
			labour	700 man—days
			land-A	0 acres
X10	Wheat production land-C	268 acres	land-B	80 acres
			Land-C	268 acres
	land-D	1042 acres		
X16	poplar production & hunting land-D	1042 acres		

SENSITIVITY ANALYSIS-GENERAL CONSIDERATIONS

Clearly, the optimum values of the variables obtained using linear programming are dependent upon the coefficients used in specifying a given problem. A relatively small change in one or more of the coefficients can affect the optimum outcome significantly. One should note that the coefficients used to define a problem nearly always vary over time. For instance, the price of white spruce being $5.60 per cord this year may change significantly next year. Secondly, the coefficients are seldom known with complete certainty; in some cases they are only approximations of what actually happens in the real world. An example of imprecise coefficients definition in the model presented in this paper is those profit coefficients which were derived from using opportunity cost as a surrogate price for outdoor recreation products. They are inaccurate because they are estimated. It should be noted also that, depending on the type of data used, models are sometimes found to be quite insensitive to changes in coefficients. By discovering such insensitive regions, one can save a lot of time and expense by using approximate figures rather than precise figures obtained through expensive studies. Still, unless all coefficients used in an application are known exactly or the sensitivity of a model and coefficient change is known to be small, it is important to investigate the effect on the optimal solution of coefficients taking on different values than the values used to obtain the initial solution.

To study the effect of different coefficient changes on the optimal solution, several variational procedures can be used. These are: (1) varying the objective function coefficients, also called profit coefficients in this paper, (2) varying the requirement coefficients, (3) varying the activity coefficients and (4) varying simultaneously the profit and requirement coefficients. Only procedures (1) and (2) were used in this study. The use of these two procedures was considered sufficient for illustrative purposes.

The range of values that a profit coefficient can assume without involving a policy change when the other coefficient are held constant is presented in Table 7. One can see that if the profit coefficient of activity X8 (multiple use of land-B) drops to $1.39 per acre, representing a change of about 3 percent, the optimum policy would be different X8 would leave the basis and X5, using land-B for white spruce production, would enter into the basis). If, indeed, the amount of uncertainty involved in estimating this coefficient is more than 3 percent, the computed or estimated optimum policy may not be the "true" optimum.

Certainly the range information provides useful data to the decision maker. For example, it indicates whether an existing policy should be altered if there is a change in market price or capital and labour costs. From a management point of view, however, the most useful interpretation of the range information is the clear definition of the alternative with which acceptable profit coefficients are associated. In particular, Krutilla (1971), among others, has emphasized the "value" of not cutting off future options. In this regard, one should see which alternative is associated with a change in profit coefficient that cuts off the fewest future options.

Along a similar line, if a linear programming solution suggests moves from forestry to agriculture involving major capital expenditures which are not considered in the model, as well as a lack of flexibility in returning to forestry, one should be wary of the optimal solution. This is particularly true if sensitivity analysis suggests that other alternatives may be pursued with nearly equal profitability. Alternatively, one could introduce into the model costs of moving to agriculture and determine a more appropriate optimal solution.

The capital expenditure needed to achieve a near optimal situation may be small and much higher to achieve an optimum so that the best solution in a linear programming sense based on a single project perspective may be poor in a broader cost-benefit or Planning Programming Budgeting (PPB) framework (Handley 1962). To expand on this point, a project can be considered in isolation or as a component of a system in which there are many priorities and goals. If a goal is to employ people, moving from an optimal linear programming solution which is not labour-intensive to a near optimal solution which is labour-intensive may make excellent sense from a systems point of view.

TABLE 7: THE RANGE OF VALUES A PROFIT COEFFICIENT CAN ASSUME, WHILE ALL OTHER COEFFICIENTS ARE HELD CONSTANT, WITHOUT INVOLVING A CHANGE OF BASIS

Activity, acre	Lower Limit, $	Original Value, $	Upper Limit, $
X1	•	- 1.86	0.12
X2	•	- 9.95	0.36
X3	•	- 1.86	0.24
X4	•	0.14	0.24
X5	•	0.24	0.28
X6	•	-73.75	2.03
X7	•	0.20	6.25
X8	1.39	1.44	infinity
X9	•	- 4.08	10.00
X10	9.69	10.00	infinity
X11	•	10.00	10.30
X12	•	- 0.08	10.06
X13	•	- 0.16	0.06
X14	•	-92.50	0.30
X15	•	- 0.16	0.72
X16	0.37	0.64	0.66

- = minus infinity

Basically, the discussion presented above suggests that linear programming is only valuable when used as a management aid (as opposed to an automated way of making decisions). Of critical importance in using linear programming is the ability of the manager to see that linear programming results are interpreted in the context of (1) alternative solutions, (2) alternative strategies that are politically and socially acceptable in the short term, and (3) the relative ease of moving to a given solution and how irreversible this move is.

SENSITIVITY ANALYSIS - THE EXAMPLE

As already noted, in the sensitivity analysis carried out, profit coefficients were allowed to vary. The lower limits of some profit coefficients were allowed to go to minus infinity since none of the activities corresponding to these coefficients was in the optimum solution. On the other hand, the fact that the upper limit of the profit coefficient of activity X8 (multiple use of land-B) is at plus infinity suggests that any increase in this profit coefficient will not effect a shift of policy, so long as the other coefficients are held constant. The upper limit of activity X10 can be similarly interpreted.

Sometimes one may want to know what it would cost to produce a unit of a non-basic activity without causing any activity to leave the basis. For instance, using one acre of land-A for white spruce production and hunting (activity X4) would decrease the total profit by $0.10 (Table 8). This would also have the effect of increasing the capital used by $1.83 and decreasing the use of land-D for poplar production and hunting (activity X16) by 0.38 acres, while activities X8 and X10 would be unaffected. The maximum number of acres that can be used to carry out activity X4, without causing any activity to leave the basis, is 145. If all of these 145 acres of land-A were used for activity X4, then total profit would decrease by $14.50 and the capital slack would be used up. Thus, in view of the small loss in total profit, it may be a "good" policy to use land-A for multiple-use rather than to leave it completely idle.

Another type of very useful information that can be gained from a post optimal analysis of a linear programming solution is that gained by varying those requirement coefficients which are used at their limit levels. If it is recalled that for the current optimal solution (Table 6) all 700 person-days of labour, 80 acres of land-B and 268 acres of land-C were used up, it may be interesting to investigate the effect on total profit, the basic activities and the other constraints if these three requirement coefficients were allowed to relax one at a time. It is seen (Table 9) that an increase of one person-day of labour increases the total profit by $1.20 and the capital used by $5.85. The $1.20 increase in total profit is the result of increasing the use of land-D by 1.89 acres for poplar production and hunting (X16). It is also seen that the increase in one person-day of labour has no effect on activities X8 and X10. On the other hand, a decrease of one person-day of labour would decrease total profit by $1.20, capital used by $5.80, land-D used by 1.89 acres, while activities X8 and X10 remain unaffected. The range of validity over which the labour constraint can be changed without causing a change of basis is 147-702 person-days. Since the original allocation of labour, 700 person-days, was entirely used up in the current optimal solution, it means that no more than two person-days can be added if the current policy of carrying out the activities X8, X10, and X16 is to be kept. However, it is possible to decrease the number of person-days by 553. Similar interpretations can be given to a unit change of the requirement coefficients of land-B and land-C.

The increased (or decreased) profit as a result of changing the coefficient of a constraint at its limit level is called marginal value. Like reduced profit, marginal value is valid only for changing one coefficient at a time within a specified range while keeping the other coefficients constant.

TABLE 8: THE EFFECT ON TOTAL PROFIT; CONSTRAINTS AT LIMIT LEVELS; AND THE BASIC VARIABLES -OF FORCING A UNIT OF A NON-BASIC VARIABLE INTO THE OPTIMAL BASIS

Activity acre	Profit $	Capital $	X8 acre	X10 acre	X16 acre	extent of substitution
X1	-1.98	2.42	*	*	-0.19	110
X2	-10.31	10.25	*	*	-0.57	26
X3	-2.10	1.83	*	*	-0.38	145
X4	-0.10	1.83.	*	*	-0.38	145
X5	-0.04	-3.68	-1	*	1.81	2
X6	-75.78	161.83	-1	*	-0.92	1
X7	-6.05	-3.64	-1	*	-7.53	79
X9	-14.08	-7.00	*	-1	*	267
X11	-0.30	-10.46	m	-1	-0.47	267
X12	-10.14	-7.29	*	-1	-0.09	267
X13	-0.22	0.21	*	*	-0.09	4
X14	-92.80	173.54	*	*	-0.47	1
X15	-0.88	-0.51	*	,	-1.13	120

* = no effect

All of the variational procedures discussed so far involved changing one coefficient at a time. Furthermore, the change of the coefficient does not cause a change of basis. However, one can also gain insight into a linear programming solution by varying several coefficients simultaneously. Such a technique is called parametric programming. (For a practical application see Handley 1962)

Of particular interest is the simultaneous variation of those profit coefficients determined by using opportunity cost (see Equation A.1 in the Appendix). One wants to ascertain the stability of this formula and thus obtain an indication of how essential its accuracy is in estimating the different coefficients of production.

TABLE 9: THE EFFECT OF A UNIT OF CHANGE OF A CONSTRAINT AT LIMIT LEVEL ON TOTAL PROFIT, THE POLICY VARIABLES, AND OTHER CONSTRAINTS

	Labour, person-day	Land-B, acre	Land-C, acre
Profit, $	1.20	0.22	9.69
Capital, $	5.85	4.39	8.54
Land-A, acre	*	*	*
Land-D, acre	1.89	-1.90	-0.47
X8, acre	* "	1.00	*
X10, acre	*	*	1.00
X16, acre	1.89	-1.90	-0.47
Range of substitution	(147,702)	(77,140)	(259,299)

* = No effect

TABLE 10: THE CHANGE OF BASIS CORRESPONDING TO A SIMULTANEOUS CHANGE OF THE OPPORTUNITY COSTS ASSOCIATED WITH RECREATION AND MULTIPLE-USE ACTIVITIES

Change in Opportunity Costs, %	Resulting Optimal Basis*	Total Profit $
1.88	(X8, X11, X16)	3,742.70
59.83	(X8, X11, X15)	14,881.60
-4.60	(X5, X10)	2,699.20

* The original optimal basis was (X8, X10, X16).

Given that the basis changed as the result of a small change of the profit coefficients of X3, X4, X7, X8, X11, X12, X15 and X16 (Table 10), it is obvious that the model is sensitive to small changes in opportunity costs. This suggests that the opportunity cost formula should be used with extreme caution. From Table 10, one also sees that when the profit coefficients change simultaneously by a mere 1.88 percent, it becomes more profitable to use land-C for fishing than for the production of wheat. The new policy as a result of the change of the profit coefficients is using land-B for white spruce and camping, land-C for fishing, and land-D for poplar and hunting. The increase in profit under the new policy is $280.70. This policy remains "optimal" until the same profit coefficients mentioned before change by 59.83 percent, when it is found more profitable to use land-D for hunting alone than for multiple use. The result of a decrease in the profit coefficients determined by using opportunity cost can be similarly interpreted.

CONCLUSION

In this paper linear programming was applied to the optimal allocation of publicly owned forestry land among different uses, some of which, such as recreation, do not have established market value. The determination of a price that should be charged the users of an outdoor recreation facility was achieved through the use of the opportunity cost concept (see the Appendix).

However, the main objective of the study was to stress the importance and usefulness of carrying out a sensitivity analysis on the input coefficients after a linear programming solution is obtained. It was emphasized that sensitivity analysis is an essential part of a linear programming application for technical reasons such as: (1) the input coefficients specified in a given problem often change (2) the input coefficients are not always exactly known, and (3) resources, such as time and money, can be saved by discovering insensitive coefficients.

Reference was also made to the specific procedures that one can employ to perform a sensitivity analysis on the input coefficients. Sensitivity analysis can provide very useful data to a decision maker on what alternative courses of action are viable and possibly better than the optimum linear progressing solution in terms of (a)preserving future options; (b) making good sense in terms of social objectives; (c) being a better solution from a systems perspective; and so on.

APPENDIX: OPPORTUNITY COST

The opportunity cost method was used to determine the price that should be charged users for the use of the land for recreation. The opportunity cost used is the value of the alternative product which would yield the maximum profit.

A.1 $P(2)=(A(1)Y(1)P(1)-A(2)Y'(1)P(1)-A(3)(C(1)+C(2)))/Y(2)$

WHERE P(2) = the price, in dollars, of one unit of a recreational product.

P(1) = the price, in dollars, of one unit of a forestry or agricultural product.

$Y(2)$ = the total number of units of a recreation product.
$Y(1)$ = the total number of units of a forestry or agricultural product.
Y'= the number of units of a forestry product in multiple use.
$C(1)$= maintenance and upkeep cost of forestry or agricultural production per acre.
$C(2)$= maintenance and upkeep cost of recreation per acre.
$A(1)$, $A(2)$, $A(3)$ = constants having the value of zero or unity.

Equation A.1 is a general equation. Depending upon whether a piece of land is used for a unique product or multiple use, three cases arise which are described below:

(a) Recreation use alone ($A(1) = A(3) = A(2) = 0$).

In this case a piece of land is used entirely for recreation and Equation A.1 becomes:

A.2 $$P(2)=((Y(1)P(1) - C(1)) + C(2))/Y(2)$$

Basically, Equation A.2 shows that the opportunity cost of using a piece of land for one unit of recreation is equal to the sum of the profit that could have been obtained if the land had been used for agricultural or forest products plus the cost of maintaining the land for recreation use, all divided by the quantity of the recreation product.

(b) Recreation use in part ($A(1) = A(2) = A(3) = 1$).

In this case, a part of a piece of land is used for recreation, the other part for forest production. Thus, Equation A.1 becomes:

(A.3) $$P(2)=(Y(1)P(1) - Y'(1)P(1) - C(1) + C(2))/Y(2)$$

The expression $Y(1)P(1) - Y'(1)P(1)$ in Equation A.3 represents that portion of the revenue foregone when only part of a piece of land is used for recreation.

(c) Double use ($A(1) = A(2) = A(3) = 0$).

Finally, a piece of land is utilized for recreation and forestry at the same time. Nothing is really foregone here, for the land is used for one production as much as for the other. The price of recreation is assumed to be at least equal to the cost of upkeep of the land for recreation. Thus, one obtains:

(A.4) $$P(2) = C(2) / Y(2)$$

Equation A.4 implies that in multiple use involving hunting or fishing, forest productivity is not diminished. This notion has also been displayed by Pichette and Potvin (see Reference 52) who, in one of their studies concerning multiple use, suggested: "It is not unreasonable to say that good forest exploitation aimed at ensuring a maximum sustained production, abundant regeneration and improved planning also promotes increased production of fauna by appropriate environmental conditions".

The use of Equations A.2, A.3, and A.4 undoubtedly requires a good knowledge of the physical capacity, productivity and cost of production of a piece of land for every possible use.

There are advantages as well as disadvantages in using opportunity cost to determine the price of outdoor recreation products. The advantages are that one would be working with timber or agricultural products for which market values are generally well established. Also, the method is relatively simple to use. An additional advantage is that it enables a decision maker to see if the cost of providing certain outdoor recreation facilities is excessively high. A disadvantage is that it only permits the definition of a minimum price for outdoor recreation products. It does not tell us anything about the value of the recreation product as perceived by the consumer. (See TN 31 and TN 38, plus references cited there.)